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ON THE BAER RINGS

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Introduction

In a recent paper [6] N.K. Thakare and S.N. Nimbhorkar give some characterizations of the compactness of the minimal spectrum of a semiprime ring with unity. These characterizations are well known in the commutative case (see, for example, A.C. Mewborn [5]).

Furthermore they give a characterization of the semiprime (weakly) Baer rings. The purpose of this paper is to prove that the (weakly) Baer rings are the complementedly normal rings whose minimal spectrum is compact. This result is known for commutative rings, see [1] second theorem. For the sake of completeness, I give the proofs, that are analogous to these of the commutative case.

Results

Let R be a semiprime (=without non zero nilpotents) ring with unity. K. Koh [3] proved that a prime ideal P is a minimal prime one if and only if $P=O_P$, where $O_P=\{x\in R \mid \exists a\notin P, ax=0\}$ coincides with the intersection of the prime ideals contained in P.

$\Sigma(R)$ denotes the set of the prime ideals of R equipped with the hull-kernel topology and $\pi(R)$ the subspace of the minimal prime ideals.

If a,b are elements of R, $ab=0$ if and only if $bRa=0$, because the ring is semiprime. Therefore we can define the so called annihilator of an element $a\in R$, by

$$a^*=\{x\in R \mid xa=0\}$$

which is a two-sided ideal.

For undefined symbols, see Thakare and Nimbhorkar [6]. In the following lemma we prove that the complementedly normal rings in the sense of J. Kist [2] are the normal rings in the sense of [6].

Lemma. *Let R be a semiprime ring with unity. The following are equivalent:*

(1) *Every prime ideal contains a unique minimal prime ideal.*

(2) *If a, b are elements of R such that $a \cdot b = 0$, then $a^* + b^* = R$.*

(3) *For every $a, b \in R$, $a^* + b^* = (ab)^*$.*

Proof. (1) *implies* (2). If $a^* + b^* \neq R$, then $a^* + b^*$ is contained in a maximal ideal M, which is prime because the ring is with unity. By the hypothesis O_M is prime and $aRb = 0$; but $a \in O_M$ implies $a^* \not\subseteq M$, which is absurd.

(2) *implies* (3). Obviously $a^* + b^* \subseteq (ab)^*$. Let $x \in (ab)^*$. Thus $(xa)b = 0$ and so there exist $z \in (xa)^*$ and $w \in b^*$ such that $1 = z + w$. Therefore $x = xz + xw$, with $xz \in a^*$ and $xw \in b^*$.

(3) *implies* (1). If P is a prime ideal, and $ab \in O_P$, there exists an element $x \in (ab)^*$ that doesn't belong to P. Therefore $x = y + z$, with $y \in a^*$ and $z \in b^*$; hence either $y \notin P$, or $z \notin P$; therefore either $a \in O_P$, or $b \in O_P$. Thus O_P is a (completely) prime ideal.

Now we can state the main result.

Theorem. *Let R be a semiprime ring with unity. The following are equivalent:*

(1) *R is a (weakly) Baer ring.*

(2) *Every prime ideal of R contains a unique minimal prime ideal and $\pi(R)$ is a compact space.*

(3) *$\pi(R)$ is a retract of $\Sigma(R)$, that is there exists a continuous function f of $\Sigma(R)$ onto $\pi(R)$ which is the identity on $\pi(R)$.*

Proof. (1) *implies* (2). If R is a Baer ring, condition (3) of Theorem 3.16 of Thakare and Nimbhorkar [6] is satisfied and so $\pi(R)$ is a compact space. Let $a, b \in R$, with $ab = 0$, and let d, e be the idempotents such that $a^* = (d)$ and $b^* = (e)$. If $a = ce$, we have $ae = ce^2 = ce = a$ so that $(1 - e) \in (d)$ and condition (2) of the lemma is satisfied.

(2) *implies* (3). Let f be the map from $\Sigma(R)$ onto $\pi(R)$ defined by $f(P) = O_P$. From the compactness condition quoted above it is enough to show that for every $a \in R$ the set $\{P \in \Sigma(R) \mid a \notin O_P\}$ is a closed set.

This is trivial because, by the definition of O_P, this set is exactly $h(a^*)$.

(3) *implies* (1). If Q is a minimal prime ideal contained in P, then $P \in \mathrm{clos}_{\Sigma(R)}\{Q\} \subseteq f^{-1}[f(Q)]$ so that $f(P) = f(Q) = Q$.

Thus for every prime P, O_P is prime and $f(P) = O_P$.

Therefore $\{P \in \Sigma(R) \mid a \notin O_P\} = h(a^*)$ is a clopen set for every $a \in R$, and so a^* is a direct summand.

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